Cost minimization with a Cobb-Douglas Function and Second Order Conditions

The production function of a company is of the Cobb-Douglas type $f(K, L) = K^{\alpha}L^{\beta}$ where $0 < \alpha < 1$ and $0 < \beta < 1$, K is capital and L labor $(K > 0, L > 0)$. Suppose that the prices of capital and labor are given by $P_k > 0$ and $P_l > 0$ respectively. Find the combination of capital and labor that minimizes the cost when the production must be Q_0 product units $(Q_0 > 0)$

Solution

The problem is to minimize $C = KP_k + LP_l$ subject to the following constraint $Q_0 = K^\alpha L^\beta$. We form the Lagrangian:

$$L = KP_k + LP_l + \lambda[Q_0 - K^\alpha L^\beta]$$

The first-order conditions are:

$$L'_K = P_k - \lambda\alpha K^{\alpha-1}L^\beta = 0$$
$$L'_L = P_l - \lambda\beta K^\alpha L^{\beta-1} = 0$$
$$L'_\lambda = Q_0 - K^\alpha L^\beta = 0$$

From the first two equations, we solve for λ:

$$\frac{P_k}{\alpha K^{\alpha-1}L^\beta} = \lambda$$
$$\frac{P_l}{\beta K^\alpha L^{\beta-1}} = \lambda$$

We equate the equations:

$$\frac{P_k}{\alpha K^{\alpha-1}L^\beta} = \frac{P_l}{\beta K^\alpha L^{\beta-1}}$$
$$\frac{P_k}{P_l} = \frac{\alpha K^{\alpha-1}L^\beta}{\beta K^\alpha L^{\beta-1}}$$

This equation shows us that the first-order conditions indicate that the slopes of the isoquants must equal while satisfying the constraint. Continuing, we solve for L:

$$\frac{P_k}{P_l}\frac{\beta}{\alpha}K = L$$

We insert this into the third first-order constraint equation:

$$Q_0 - K^\alpha[\frac{P_k}{P_l}\frac{\beta}{\alpha}K]^\beta = 0$$

$$Q_0 - K^{\alpha+\beta}[\frac{P_k}{P_l}\frac{\beta}{\alpha}]^\beta = 0$$

Solving for K:

$$Q_0[\frac{P_l}{P_k}\frac{\alpha}{\beta}]^\beta = K^{\alpha+\beta}$$

$$\left[Q_0[\frac{P_l}{P_k}\frac{\alpha}{\beta}]^\beta\right]^{\frac{1}{\alpha+\beta}} = K^*$$

Now we insert into the function for L:

$$\frac{P_k}{P_l}\frac{\beta}{\alpha}K^* = L$$

$$\frac{P_k}{P_l}\frac{\beta}{\alpha}\left[Q_0[\frac{P_l}{P_k}\frac{\alpha}{\beta}]^\beta\right]^{\frac{1}{\alpha+\beta}} = L$$

Simplifying:

$$Q_0^{\frac{1}{\beta+\alpha}}\frac{P_k}{P_l}\frac{\beta}{\alpha}\left[\frac{P_k}{P_l}\frac{\beta}{\alpha}\right]^{\frac{-\beta}{\beta+\alpha}} = L$$

$$Q_0^{\frac{1}{\beta+\alpha}}\left[\frac{P_k}{P_l}\frac{\beta}{\alpha}\right]^{\frac{\alpha}{\beta+\alpha}} = L$$

$$\left[Q_0[\frac{P_k}{P_l}\frac{\beta}{\alpha}]^\alpha\right]^{\frac{1}{\beta+\alpha}} = L^*$$

Now let's solve the second-order conditions: First, we calculate the second derivatives:

$$L''_{KK} = -\lambda\alpha(\alpha-1)K^{\alpha-2}L^{\beta}$$

$$L''_{LL} = -\lambda\beta(\beta-1)K^{\alpha}L^{\beta-2}$$

$$L''_{LK} = -\lambda\beta\alpha K^{\alpha-1}L^{\beta-1}$$

$$L''_{KL} = -\lambda\beta\alpha K^{\alpha-1}L^{\beta-1}$$

Now the derivatives corresponding to the bordered Hessian:

$$g'_K = \alpha K^{\alpha-1}L^{\beta}$$

$$g'_L = \beta K^{\alpha}L^{\beta-1}$$

To meet the second-order conditions, we construct the bordered Hessian. If we are at a minimum, the determinant of the bordered Hessian must be negative.

$$\bar{H} = \begin{pmatrix} 0 & g'x & g'y \\ g'x & L''_{xx} & L''_{xy} \\ g'y & L''_{yx} & L''_{yy} \end{pmatrix} = \begin{pmatrix} 0 & \alpha K^{\alpha-1}L^{\beta} & \beta K^{\alpha}L^{\beta-1} \\ \alpha K^{\alpha-1}L^{\beta} & -\lambda\alpha(\alpha-1)K^{\alpha-2}L^{\beta} & -\lambda\beta\alpha K^{\alpha-1}L^{\beta-1} \\ \beta K^{\alpha}L^{\beta-1} & -\lambda\beta\alpha K^{\alpha-1}L^{\beta-1} & -\lambda\beta(\beta-1)K^{\alpha}L^{\beta-2} \end{pmatrix}$$

Before replacing in the optimal values, we calculate the determinant of the bordered Hessian:

$$-\alpha K^{\alpha-1}L^{\beta} \begin{vmatrix} \alpha K^{\alpha-1}L^{\beta} & \beta K^{\alpha}L^{\beta-1} \\ -\lambda\beta\alpha K^{\alpha-1}L^{\beta-1} & -\lambda\beta(\beta-1)K^{\alpha}L^{\beta-2} \end{vmatrix}$$
$$+\beta K^{\alpha}L^{\beta-1} \begin{vmatrix} \alpha K^{\alpha-1}L^{\beta} & \beta K^{\alpha}L^{\beta-1} \\ -\lambda\alpha(\alpha-1)K^{\alpha-2}L^{\beta} & -\lambda\beta\alpha K^{\alpha-1}L^{\beta-1} \end{vmatrix}$$

We calculate the first term:

$$-\alpha K^{\alpha-1}L^{\beta}\left[(\alpha K^{\alpha-1}L^{\beta})(-\lambda\beta(\beta-1)K^{\alpha}L^{\beta-2}) - (\beta^{\alpha}L^{\beta-1})(-\lambda\beta\alpha K^{\alpha-1}L^{\beta-1})\right]$$

Simplifying the negatives and remembering that $\beta < 1$

$$\underbrace{-\alpha K^{\alpha-1}L^{\beta}\underbrace{\left[\underbrace{(\alpha K^{\alpha-1}L^{\beta})(-\lambda\beta\underbrace{(\beta-1)}_{-}K^{\alpha}L^{\beta-2})}_{+} + (\beta^{\alpha}L^{\beta-1})(\lambda\beta\alpha K^{\alpha-1}L^{\beta-1})\right]}_{+}}_{-}$$

This term is even before inserting the optimal values of K, L, or λ since these three terms at the optimum are positive and do not affect the previous conclusion. Now we calculate the second term of the determinant:

$$+\beta K^{\alpha}L^{\beta-1}\left[\alpha K^{\alpha-1}L^{\beta}(-\lambda\beta\alpha K^{\alpha-1}L^{\beta-1}) - \beta K^{\alpha}L^{\beta-1}(-\lambda\alpha(\alpha-1)K^{\alpha-2}L^{\beta})\right]$$

Simplifying the negatives:

$$\underbrace{+\beta K^{\alpha}L^{\beta-1}\left[\underbrace{-\alpha K^{\alpha-1}L^{\beta}(\lambda\beta\alpha K^{\alpha-1}L^{\beta-1})}_{-} + \underbrace{\beta K^{\alpha}L^{\beta-1}(\lambda\alpha\underbrace{(\alpha-1)}_{-}K^{\alpha-2}L^{\beta})}_{-}\right]}_{-}$$

If $\alpha - 1 < 0$, then we have that the determinant of the bordered Hessian is negative, and we are at a minimum.